Exercise 2E

1 a $\bar{x}=\frac{\sum x}{n}=\frac{143}{100}=1.43$

$\sigma^2=\frac{\sum x^2}{n}-(\bar{x})^2=\frac{347}{100}-1.43^2=1.4251$

b Both the sample mean and the sample variance equal 1.43 correct to three significant figures. The mean is close in value to the variance, which supports the choice of a Poisson distribution as a model.

c Estimate $\lambda=1.4$ from the data, so $X\sim \text{Po}(1.4)$

$\text{P}(X=3)=\frac{e^{-1.4}\times 1.4^3}{3!}=0.1128 \text{ (4 d.p.)}$

2 a Let X be the number of cars passing the checkpoint in a 5-minute period.

$\bar{x}=\frac{\sum fx}{\sum f}=\frac{7\times 0+21\times 1+30\times 2+41\times 3+36\times 4+29\times 5+21\times 6+11\times 7+4\times 8}{7+21+30+41+36+29+21+11+4}=3.64$

$\sigma^2=\frac{\sum fx^2}{\sum f}-(\bar{x})^2$

$=\frac{7\times 0^2+21\times 1^2+30\times 2^2+41\times 3^2+36\times 4^2+29\times 5^2+21\times 6^2+11\times 7^2+4\times 8^2}{7+21+30+41+36+29+21+11+4}-3.64^2$

$=3.5604$

b Both the sample mean and the sample variance equal 3.6 correct to one decimal place. The mean is close in value to the variance, which supports the choice of a Poisson distribution as a model.

c Estimate $\lambda=3.6$ from the data, so $X\sim \text{Po}(3.6)$. Required value must be found using a calculator:
$\text{P}(X\leqslant 2)=0.3027 \text{ (4 d.p.)}$

d From the data, the relative frequency of obtaining no more than 2 cars in a 5-minute period is:
$\frac{7+21+30}{200}=\frac{58}{200}=0.29$
which is close to the value calculated in part **c** and so further supports the choice of model.

3 a Let X be the number of flaws in a piece of cloth.

$\bar{x}=\frac{\sum fx}{\sum f}=\frac{8\times 0+19\times 1+28\times 2+25\times 3+19\times 4+11\times 5+7\times 6+3\times 7}{8+19+28+25+19+11+7+3}=2.867 \text{ (3 d.p.)}$

$\sigma^2=\frac{\sum fx^2}{\sum f}-(\bar{x})^2$

$=\frac{8\times 0^2+19\times 1^2+28\times 2^2+25\times 3^2+19\times 4^2+11\times 5^2+7\times 6^2+3\times 7^2}{8+19+28+25+19+11+7+3}-2.8667^2$

$=2.899 \text{ (3 d.p.)}$

3 b Both the sample mean and the sample variance equal 2.9 correct to one decimal place. The fact that the mean is close in value to the variance supports the choice of a Poisson distribution as a model for flaws in a piece of cloth.

c The sample of only 120 pieces of cloth will be unreliable for estimating a proportion (especially of extreme values) for a much larger sample. The fact that all of the 120 pieces sampled had fewer than 8 flaws should not be taken as meaning that it is impossible, just rare, for a piece of cloth to have 8 or more flaws and scaling up to give an estimate of 0 out of 10 000 would not be advisable.

d Estimate $\lambda = 2.9$ from the data, so $X \sim \text{Po}(2.9)$. Required value must be found using a calculator:

$$\begin{aligned}
\text{P}(X \geqslant 8) &= 1 - \text{P}(X \leqslant 7) \\
&= 1 - 0.9901 \\
&= 0.0099 \text{ (4 d.p.)}
\end{aligned}$$

So there would be an estimated 99 pieces with at least 8 flaws.

Challenge

$\text{E}(X) = \sum x\text{P}(X = x)$	definition of $\text{E}(X)$
$= \sum_{x=0}^{\infty} x \dfrac{\text{e}^{-\lambda}\lambda^x}{x!}$	given distribution
$= 0 + \sum_{x=1}^{\infty} x \dfrac{\text{e}^{-\lambda}\lambda^x}{x!}$	splitting off first term (0)
$= \text{e}^{-\lambda} \sum_{x=1}^{\infty} \dfrac{x\lambda^x}{x!}$	taking out factor $\text{e}^{-\lambda}$
$= \text{e}^{-\lambda} \sum_{x=1}^{\infty} \lambda \dfrac{\lambda^{x-1}}{(x-1)!}$	cancelling common factor x in summation
$= \text{e}^{-\lambda}\lambda \sum_{r=0}^{\infty} \dfrac{\lambda^r}{r!}$	relabelling with $r = x - 1$
$= \text{e}^{-\lambda}\lambda(\text{e}^{\lambda})$	known expansion $\text{e}^{\lambda} = \sum \dfrac{\lambda^r}{r!}$
$= \lambda$	

Challenge (continued)

$\text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$	definition of Var(X)
$= \sum x^2 \text{P}(X = x) - (\text{E}(X))^2$	definition of $\text{E}(X^2)$
$= \sum_{x=0}^{\infty} x^2 \frac{\text{e}^{-\lambda}\lambda^x}{x!} - \lambda^2$	given distribution
$= 0 + \sum_{x=1}^{\infty} x^2 \frac{\text{e}^{-\lambda}\lambda^x}{x!} - \lambda^2$	splitting off first term
$= \text{e}^{-\lambda} \sum_{x=1}^{\infty} \frac{(x^2 - x + x)\lambda^x}{x!} - \lambda^2$	taking out factor $\text{e}^{-\lambda}$ and rearranging
$= \text{e}^{-\lambda} \sum_{x=1}^{\infty} \left(\frac{x(x-1)\lambda^x}{x!} + \frac{x\lambda^x}{x!} \right) - \lambda^2$	expanding
$= \text{e}^{-\lambda} \left(\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) - \lambda^2$	cancelling common factors summation terms
$= \text{e}^{-\lambda} \left(\lambda^2 \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} + \lambda \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \right) - \lambda^2$	relabelling with $s = x - 2,\ r = x - 1$
$= \text{e}^{-\lambda}(\lambda^2 \text{e}^{\lambda} + \lambda \text{e}^{\lambda}) - \lambda^2$	known expansion $\text{e}^{\lambda} = \sum \frac{\lambda^r}{r!}$
$= \lambda$	